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# EFFECT OF THE DIURNAL ATMOSPHERIC BULGE ON SATELLITE ACCELERATIONS

by

Stanley P. Wyatt

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Summary. --Formulas are developed to express the secular acceleration of a satellite on passing through an atmosphere which bulges in the sunward direction and in which the scale height increases with height, these two properties of the high atmosphere having previously been established from satellite observations. Comparison of the new formulas with those for a spherically symmetric atmosphere of constant scale height indicates that deduced atmospheric densities may be systematically incorrect by up to 50 or 60 percent at heights of 500 to 600 km when the earlier and simpler equations are used.

## 1. The Spherically Symmetric Isothermal Atmosphere

Several authors have derived expressions for the secular acceleration of a satellite moving through the high atmosphere (Sterne, 1958; Groves, 1958; King-Hele, Cook, and Walker, 1959). The simplest and most tractable assumptions one can make are that the terrestrial atmosphere is spherically symmetric, stationary, and of constant scale height throughout those strata travelled by the satellite in question. Within the framework of these assumptions, the secular acceleration is given by

$$\frac{\Delta P}{P} = -3C_D \frac{A}{m} \frac{q\rho_q}{(1-e)} \int_0^{\pi} \frac{(1 + e\cos E)^{3/2}}{(1 - e\cos E)^{1/2}} \epsilon^{-c(1 - \cos E)} \, dE \,. \quad (1)$$

Here, $\Delta P/P$ is the dimensionless change of period per period; $C_D$ the drag coefficient; $A/m$ the ratio of the average geometrical cross-section of the satellite to its mass; $q$ the perigee distance and $e$ the eccentricity of the satellite; $\rho_q$ the atmospheric density at the level of perigee; $E$ the eccentric anomaly of the satellite; and $\epsilon$ the base of natural logarithms. The dimensionless constant $c$ is defined by $c = qe/H(1 - e)$, where $H$ is the scale height of the atmosphere and is assumed to be constant.

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The product of these factors is

$$\sqrt{\frac{2H}{qe}}\,(1 + e)^{3/2}\,dy\left[1 + \frac{H}{q}My^2 + \frac{H^2}{q^2}Ny^4 + \ldots\right], \tag{28}$$

where M and N, given by equations (3), are functions of e alone. Physically, these terms occur because atmospheric drag exerts a tangential perturbing force; they appear independently of any particular choice of atmospheric model. Third, we need to express $\psi'$ in terms of y. We have

$$\cos^6(\psi'/2) = \frac{1}{8}(1 + \cos\psi')^3 .$$

Substitution of equation (18) then gives us an equation for $\psi'$ as a function of E. With the transformation equation (25) we then obtain $\psi'$ as a power series in $\sqrt{(H/q)}\ y$. Finally, note that the upper limit of integration of equation (13) becomes

$$y^2 = \frac{2qe}{H(1 - e)} .$$

Since we are here dealing with orbits of appreciable eccentricity we may extend the upper limit to infinity without appreciable error. All integrations involving odd powers of y then vanish, and therefore $\psi'$ can be expressed for our purposes as a power series in $(H/q)\,y^2$. The formula turns out to be

$$\cos^6(\psi'/2) = u - v\frac{H}{q}y^2 + w\frac{H^2}{q^2}y^4 - \ldots . \tag{29}$$

The coefficients u, v, and w are constants for a given orbit; they are given by

$$\begin{aligned}
u &= \frac{1}{8}(1 + \mu)^3 = \cos^6(\psi_q'/2),\\
v &= \frac{3(1 + e)}{8e}(1 + \mu)\left[\mu(1 + \mu) - 2\nu^2\right],\\
w &= \frac{3(1 + e)}{8e^2}\left[\mu(1 + \mu)\{\mu + 2e\mu + e\} - \nu^2\{1 + 3\mu + 5e\mu + 3e\}\right].
\end{aligned} \tag{30}$$

where a is the orbital semimajor axis and where

$$V_o = 1 + \frac{3}{2}e^2 - \frac{e^3}{c}, \tag{6a}$$

and

$$V_1 = 2 - \frac{3}{2}\frac{e}{c} + e^2 \; 1 + \frac{2}{c^2} \; . \tag{6b}$$

Tables of $\epsilon^{-c} I_o(c)$ and $\epsilon^{-c} I_1(c)$ are available in Watson's (1944) treatise on Bessel functions. For a circular orbit, equation (5) reduces to

$$\frac{\Delta P}{P} = - 3\pi C_D \frac{A}{m} a\rho_a \, . \tag{7}$$

2. Some Complicating Effects

Modifications of these equations have been derived for several effects. One interesting generalization is the assumption that atmospheric density is a function of height above the oblate-spheroidal earth rather than above a sphere (Groves, 1958; Sterne, 1959). Thus when the perigee of a satellite is located at high latitude, the predicted drag is less than when it is later located near the equator. For a polar satellite, the secular acceleration arising from such an effect would vary approximately 15 percent either side of the mean during rotation of perigee for a scale height of 50 km, and approximately 10 percent for a scale height of 100 km. For low-inclination satellites, the variation would be less. As we shall see later, however, this effect is masked by others for high satellites.

A second modification of the drag equation takes account of the rotation of the atmosphere with the earth (Sterne, 1959). A satellite moving in a direct orbit experiences a "headwind" of smaller magnitude than does one moving in a retrograde orbit. For equatorial orbits, neglect of atmospheric rotation leads to errors of approximately 10 percent in the secular accelerations. For orbits of higher inclination the error is smaller.

A third modification accounts for the increasing scale height of the atmosphere with height above ground (Jacchia, 1960b). From data at heights near 400 km, Jacchia finds that neglect of this variation leads to overestimates of the acceleration by approximately 5 percent in the nighttime atmosphere, and up to 10 percent in the daytime atmosphere. The maximum error occurs near e = 0.02; of course, the error must be zero for a circular orbit.

3. The Diurnal Bulge of the Atmosphere

The assumption of a spherically symmetric atmosphere at heights of several hundreds of kilometers has been shown to be untenable by several analyses (Jacchia, 1959; 1960a; Priester and Martin, 1960; Wyatt, 1959). The high atmosphere bulges toward a point in the sky some 15° to 30° east of the sun. For a fixed index of solar activity, the observed accelerations of Satellite 1958 β2 (Vanguard I) indicate that the air density at 665 km is about ten times as great when perigee passage occurs an hour or two after noon as when it occurs during the night. It is also clear from observations of several satellites that the scale height of the atmosphere increases with height at all times of day.

Interpreting these results physically, Nicolet (1960) finds that the density of the high atmosphere is governed largely by the absorption of solar ultraviolet radiation below 200 km. The influx of energy fixes the temperature gradient of the atmosphere between 200 and 300 km and also the temperature of the nearly isothermal atmosphere at greater heights. Because the ultraviolet input at any place depends both on the time of day there and on the general level of solar activity, the high-atmosphere density variations depend on both of these parameters, as observed. Above approximately 300 km the time scale of heat conduction is short and any vertical column of air is isothermal, provided that the injection of heat from the Chapman corona is small. Although T is presumably constant at these levels, the scale height, $H = kT/g\overline{m}$, increases with height. The acceleration of gravity decreases, of course, as the inverse square of the distance from the earth's center, thus contributing to the increase of H. The mean molecular weight, $\overline{m}$, also decreases with height, because above approximately 150 km the air is in diffusion equilibrium; each type of molecule is sorted out according to its mass and thus the concentration of $N_2$ relative to O decreases with height. This factor contributes importantly to the observed increase of scale height with height. It should be added that, although a vertical column above 300 km is spatially isothermal at any one moment, there is a temporal variation of about 500°K between day and night, a variation arising from the varying injection of solar radiant energy at the lower atmospheric levels.

## 4. The Fundamental Drag Equation in a Bulging Atmosphere

Jacchia (1960a) stresses that the diurnal bulge will distort satellite motions from the motions they would have if the atmosphere were spherically symmetric. The chief problem addressed in the present paper is the derivation of a fundamental drag equation for a bulging atmosphere in which the scale height increases with height, followed by a comparison with the spherical approximation with H constant. At the expense of some added calculation, the new equation should permit the derivation of more precise atmospheric densities and other parameters of the high atmosphere.

The most tractable assumption is that the atmosphere is axially symmetric, and this assumption is not at odds with observations to date. I shall adopt it here and, with Jacchia (1960a), shall further assume that the atmospheric bulge always points toward the same declination as the sun, but with a lag angle $\lambda$. Thus the right ascension of the symmetry axis is $\alpha_{\odot} + \lambda$. Previous analyses indicate that $15° \leqq \lambda \leqq 30°$, so that at any given level of the high atmosphere the peak density occurs between one and two hours after local noon. Jacchia's analysis of the accelerations of Satellites 1958 Alpha, 1958 $\beta 2$, 1958 $\delta 2$, and 1959 $\alpha 1$ shows that between 200 km and 700 km the atmospheric density can be well represented by

$$\rho = \rho_o(z) F_{20} \left\{ 1 + 0.19 \left[ \exp(0.0055z) - 1.9 \right] \cos^6 (\psi'/2) \right\} , \tag{8}$$

where

$$\log \rho_o(z) = - 16.021 - 0.001985z + 6.363 \exp(-0.0026z) . \tag{9}$$

These equations are in cgs units except that z, the height above ground, is in kilometers. The angle $\psi'$ is the geocentric angle from the axis of the bulge. Equation (9) gives the density as a function of height for the nighttime atmosphere at a moment when the daily-mean 20-cm solar flux of $F_{20}$ (Priester and Martin, 1960) is unity. Unit flux density is defined as $10^{-20}$ W $m^{-2}(c/s)^{-1}$. As can be seen from equation (8), Jacchia finds that the density at a given point is proportional to the first power of the solar flux, other factors being equal. In the brackets of equation (8), the dependence on $\psi'$ indicates the sharpness of the bulge as found empirically and the dependence on z shows that the diurnal effect increases with height. Equation (8) is of such a form that it includes the effect of increasing scale height with height. Jacchia's two formulas are remarkably successful in representing satellite observations to date. Although they are empirical in nature, I propose to adopt them as a basis for computing the drag equation in a bulging atmosphere.

For any satellite in an orbit of moderate eccentricity, the maximum drag occurs not necessarily at perigee passage, but nevertheless fairly near it. For another satellite in a more circular orbit, the maximum drag may occur at any value of the true anomaly. In either case, the height above perigee, $z - z_q$, at which significant drag occurs is never very large. Let us therefore express equations (8) and (9) in terms of $s = z - z_q$. Also, for economy we shall deduce the drag equation for a constant level of solar activity, setting $F_{20} = 1$. Equation (9) for the nighttime atmosphere, $\Psi' = \pi$, becomes

$$\rho_o(s) \equiv \rho(s, \pi) = \rho(0, \pi) \epsilon^{-.004571s - 14.651 \epsilon^{-.0026 z_q} (1 - \epsilon^{.0026s})}$$

$$\cong \rho(0, \pi) \epsilon^{-\left(.004571 + .03809 \epsilon^{-.0026 z_q}\right) s} \left(1 + .00004952 \epsilon^{-0.0026 z_q} s^2\right). \tag{10}$$

The approximation here introduced is needed in order for us to integrate the equations. It underestimates the density given in equation (9) by about one percent two scale heights above perigee and by about 20 percent four scale heights above perigee in the range 200 km $\leqq z_q \leqq$ 600 km; thus, it is an adequate representation for any given orbit. Equation (8) becomes, without approximation,

$$\rho \equiv \rho(s, \Psi') = \rho(s, \pi) \left\{ 1 + 0.19 \left[ \epsilon^{.0055 z_q} \epsilon^{.0055s} - 1.9 \right] \cos^6 (\Psi'/2) \right\}. \tag{11}$$

The general density function we shall employ is obtained by substituting equation (11) in (10) and is

$$\rho(s, \Psi') = \rho(0, \pi) \left\{ \left[ 1 - K \cos^6 (\Psi'/2) \right] \epsilon^{-Qs} + L \cos^6 (\Psi'/2) \epsilon^{-Rs} \right\} \left(1 + Us^2\right), \tag{12}$$

where

$$K = .361,$$

$$L = .19 \epsilon^{.0055 z_q},$$

$$Q = (.03809 \epsilon^{-.0026 z_q} + .004571) \text{ km}^{-1},$$

$$R = (.03809 \epsilon^{-.0026 z_q} - .000929) \text{ km}^{-1},$$

$$U = .00004952 \epsilon^{-.0026 z_q} \text{ km}^{-2}.$$

These quantities are either constant or are functions of perigee height alone and can be tabulated once and for all. We shall consider the physical significance of some of them later.

Having chosen equation (12) for the density function, we find that the secular acceleration of a satellite becomes

$$\frac{\Delta P}{P} = -3C_D \frac{A}{m} \frac{q\rho(0,\pi)}{(1-e)} \int_0^{\pi} \frac{(1+e\cos E)^{3/2}}{(1-e\cos E)^{1/2}}$$

$$\times \left\{ \left[1 - K\cos^6(\psi'/2)\right] \epsilon^{-Qs} + L\cos^6(\psi'/2)\epsilon^{-Rs} \right\} (1+Us^2)dE. \tag{13}$$

To progress further, we must express both $\psi'$ and s as functions of E, the eccentric anomaly.

First let us find the relation between $\psi'$, the instantaneous geocentric angle between the bulge-axis and the satellite, and E. Figure 1 shows the geometry, with A the north celestial pole, V the vernal equinox, N the ascending node of the orbit, Q the direction of perigee, P the instantaneous position of the satellite, and B the instantaneous direction of the atmospheric bulge. We wish to evaluate $\psi' \equiv \widehat{BP}$. In triangle ABP we have

$$\cos\psi' = \sin\delta_B \sin\delta_P + \cos\delta_B \cos\delta_P \cos(\alpha_P - \alpha_B), \tag{14}$$

where $\alpha_B$, $\delta_B$; $\alpha_P$, $\delta_P$ are the right ascension and declination of the bulge and satellite respectively. By hypothesis $\alpha_B = \alpha_\odot + \lambda$, $\delta_B = \delta_\odot$; and since these coordinates change slowly we shall regard them as effectively fixed over an interval of a few days. Next let us express the equatorial coordinates of the satellite as functions of its slowly varying orbital elements and of its rapidly varying eccentric anomaly. In the spherical triangle NFP, we have $\widehat{NF} = \alpha_P - \alpha_N$, where $\alpha_N$ is the right ascension of the ascending node; FP = $\delta_P$; $\widehat{NP} = \omega + \theta$, where $\omega$ is the argument of perigee and $\theta$ is the true anomaly; $\angle$FNP = i, the inclination; and $\angle$NFP = 90°. With the aid of the relations

$$\begin{aligned} \sin\delta_P &= \sin i \sin(\omega+\theta), \\ \cos(\omega+\theta) &= \cos\delta_P \cos(\alpha_P - \alpha_N), \\ \tan(\alpha_P - \alpha_N) &= \cos i \tan(\omega+\theta), \end{aligned} \tag{15}$$

the desired formula comes out to be

$$\cos\psi' = \mu\cos\theta + \nu\sin\theta,$$

$$\mu = \sin\delta_B \sin i \sin\omega + \cos\delta_B\left[\cos(\alpha_N - \alpha_B)\cos\omega - \cos i \sin(\alpha_N - \alpha_B)\sin\omega\right], \tag{16}$$

$$\nu = \sin\delta_B \sin i \cos\omega - \cos\delta_B\left[\cos(\alpha_N - \alpha_B)\sin\omega + \cos i \sin(\alpha_N - \alpha_B)\cos\omega\right].$$

Next, we define the density scale height (Jacchia, 1960b) by

$$\frac{1}{H(0,\psi'_q)} \equiv - \frac{1}{\rho}\frac{\partial \rho}{\partial s}\bigg|_{0,\psi'_q} . \tag{23}$$

The density scale height is identical with the ordinary pressure scale height, $kT/g\overline{m}$, if the temperature gradient and molecular-mass gradients are zero separately or if $T/\overline{m}$ is independent of height. Under actual conditions, with $\partial T/\partial s \geqq 0$ and $\partial \overline{m}/\partial s \leqq 0$, the density scale height is somewhat smaller than the ordinary scale height. Differentiating equation (12) and evaluating the right side of equation (23) at $s = 0$, $\psi' = \psi'_q$, we obtain

$$H(0,\psi'_q) = \frac{\{1 + (L - K)\cos^6(\psi'_q/2)\}}{Q\{1 - K\cos^6(\psi'_q/2) + RL\cos^6(\psi'_q/2)\}} . \tag{24}$$

In particular, we may now interpret the constant Q, since equation (24) shows that $H(0,\pi) = Q^{-1}$. Thus Q is the reciprocal of the nighttime scale height at perigee. The perigee scale height on the bulge axis is $H(0,0) = \{1 + (L - K)\}\{Q(1 - K) + RL\}^{-1}$.

## 5. The Equation for an Eccentric Orbit

Let us define a dimensionless variable of integration, $y^2$, by $y^2 \equiv s/H$, where for simplicity in the sequel we shall write $H(0,\psi'_q) \equiv H$. It follows from equation (21) that

$$1 - \cos E = \frac{H(1 - e)}{qe} y^2 . \tag{25}$$

This is the same substitution effected in section 1, but it should be stressed that we are here concerned with the scale height at a perigee point located at a particular height and at a particular angular distance from the sunward bulge. Our next task is to transform the integrand of equation (13) into a function of y alone. First, the three factors involving s transform very simply, and exactly, by substitution of $s = Hy^2$. Second, the three factors explicitly containing E may be expressed as power series in $(H/q)y^2$. In particular,

$$\frac{(1 + e\cos E)^{3/2}}{(1 - e\cos E)^{1/2}} = \frac{(1 + e)^{3/2}}{(1 - e)^{1/2}}\left[1 - \frac{H(2 - e)}{q(1 + e)} y^2 + \frac{3H^2}{2q^2(1 + e)^2} y^4 - \ldots\right] , \tag{26}$$

and

$$dE = \sqrt{\frac{2H(1 - e)}{qe}}\left[1 + \frac{H(1 - e)}{4qe} y^2 + \frac{3H^2(1 - e)^2}{32q^2e^2} y^4 + \ldots\right] dy . \tag{27}$$

Next, the true anomaly is related to the eccentric anomaly and e, the eccentricity, by the equations

$$\cos\theta = \frac{\cos E - e}{1 - e\cos E}, \qquad \sin\theta = \frac{\sqrt{1-e^2}\,\sin E}{1 - e\cos E}. \tag{17}$$

It thus turns out that the required angular separation of the satellite and the bulge axis is

$$\cos\psi' = \frac{\mu(\cos E - e) + \nu\sqrt{1-e^2}\,\sin E}{1 - e\cos E}. \tag{18}$$

An alternative formula for $\psi'$ can be found if we consider in figure 1 the triangle BQP. We have

$$\cos\psi' = \cos\psi'_q \cos\theta + \sin\psi'_q \cos\chi \sin\theta, \tag{19}$$

where $\psi'_q = \widehat{QB}$ is the geocentric angle between perigee and the bulge axis; $\chi = \angle BQP$; and $\theta$ again is the true anomaly. Comparison with equations (16) shows that

$$\mu = \cos\psi'_q, \qquad \nu = \sin\psi'_q \cos\chi. \tag{20}$$

We next wish to obtain s as a function of E. We readily find that

$$s = z - z_q = r - q = a(1 - e\cos E) - a(1 - e) = \frac{qe(1 - \cos E)}{1 - e}. \tag{21}$$

Substitution of equations (18) and (21) in equation (13) yields the equation for the secular acceleration of a satellite passing through a Jacchia-type atmosphere, the integrand now being a function of E and of miscellaneous "constants" of the atmosphere and the satellite orbit.

Finally, instead of expressing the acceleration in terms of the nighttime density, $\rho(0,\pi)$, we find it more useful to give it as a function of the density and scale height at the perigee point at the time in question, since the latter quantities are the ones that can be deduced from the observations. From equation (12) it is found that

$$\rho(0,\psi'_q) = \rho(0,\pi)\left\{1 + (L - K)\cos^6(\psi'_q/2)\right\}. \tag{22}$$

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The density scale height is identical with the ordinary pressure scale height, $kT/g\overline{m}$, if the temperature gradient and molecular-mass gradients are zero separately or if $T/\overline{m}$ is independent of height. Under actual conditions, with $\partial T/\partial s \geqq 0$ and $\partial \overline{m}/\partial s \leqq 0$, the density scale height is somewhat smaller than the ordinary scale height. Differentiating equation (12) and evaluating the right side of equation (23) at $s = 0$, $\psi' = \psi'_q$, we obtain

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where $\psi_q' = \widehat{QB}$ is the geocentric angle between perigee and the bulge axis; $\chi = \angle BQP$; and $\theta$ again is the true anomaly. Comparison with equations (16) shows that

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The product of these factors is

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where M and N, given by equations (3), are functions of e alone. Physically, these terms occur because atmospheric drag exerts a tangential perturbing force; they appear independently of any particular choice of atmospheric model. Third, we need to express $\psi'$ in terms of y. We have

$$\cos^6(\psi'/2) = \frac{1}{8}(1 + \cos\psi')^3 .$$

Substitution of equation (18) then gives us an equation for $\psi'$ as a function of E. With the transformation equation (25) we then obtain $\psi'$ as a power series in $\sqrt{(H/q)}\ y$. Finally, note that the upper limit of integration of equation (13) becomes

$$y^2 = \frac{2qe}{H(1 - e)} .$$

Since we are here dealing with orbits of appreciable eccentricity we may extend the upper limit to infinity without appreciable error. All integrations involving odd powers of y then vanish, and therefore $\psi'$ can be expressed for our purposes as a power series in $(H/q)\,y^2$. The formula turns out to be

$$\cos^6(\psi'/2) = u - v\frac{H}{q}y^2 + w\frac{H^2}{q^2}y^4 - \ldots . \tag{29}$$

The coefficients u, v, and w are constants for a given orbit; they are given by

$$u = \frac{1}{8}(1 + \mu)^3 = \cos^6(\psi'_q/2) ,$$

$$v = \frac{3(1 + e)}{8e}(1 + \mu)\left[\mu(1 + \mu) - 2\upsilon^2\right] , \tag{30}$$

$$w = \frac{3(1 + e)}{8e^2}\left[\mu(1 + \mu)\{\mu + 2e\mu + e\} - \upsilon^2\{1 + 3\mu + 5e\mu + 3e\}\right] .$$

When we substitute equations (21), (22), (25), (28), and (29) in equation (13), the secular acceleration becomes

$$\frac{\Delta P}{P} = - \frac{3\sqrt{\pi}}{\sqrt{2}} C_D \frac{A}{m} \frac{\sqrt{Hq}}{\sqrt{e}} \frac{(1 + e)^{3/2}}{(1 - e)} \frac{\rho(0, \psi'_q)}{\{1 + u(L - K)\}}$$

$$\times \int_0^\infty \frac{2}{\sqrt{\pi}} dy \left[1 + (H/q) My^2 + (H^2/q^2) Ny^4 + \ldots\right]$$

$$\times \left[\epsilon^{-QHy^2} \{1 - Ku + (H/q) Kvy^2 - (H^2/q^2) Kwy^4 - \ldots\} \right. \tag{31}$$

$$\left. + \epsilon^{-RHy^2} \{Lu - (H/q) Lvy^2 + (H^2/q^2) Lwy^4 - \ldots\}\right]$$

$$\times \left[1 + UH^2y^4\right].$$

Equation (31) can be integrated term by term after multiplication of the several brackets. Note that the three power series in the first two brackets of the integrand have been truncated at $(H^2/q^2) y^4$ in order to keep the calculation manageable. The coefficients of terms of higher order decrease approximately in the ratio H/qe, which is less than about 0.1 provided the eccentricity is not very small. The justification for terminating the final bracket at order $y^4$ was described in section 4.

## 6. Comparison with the Simpler Drag Formula

Instead of writing down the general result of integrating equation (31), it is simpler to give the ratio of this acceleration to that given by equation (2) for the spherically symmetric atmosphere of constant scale height. One reason, of course, is that the coefficients are nearly identical. A second reason is that the ratio is independent of $F_{20}$, the solar flux at 20 cm. Let us call this ratio J. To state its meaning more precisely, consider a date on which the perigee of a satellite lies $z_q$ km above the earth's surface and at a geocentric angle $\psi'_q$ from the bulge. The quantity J is then the secular acceleration of the satellite as it describes one orbit through the Jacchia atmosphere divided by the secular acceleration of the same satellite as it describes the same orbit through a spherically symmetric atmosphere in which the scale height is constant and in which the density and scale height agree with the Jacchia atmosphere at the perigee point (but not necessarily anywhere else along the orbit).

I have integrated equation (31) and computed values of J for perigee heights ranging from 200 km to 600 km, for eccentricities from 0.1 to 0.3, and for four specific orientations of the orbit with respect to the bulge. It turns out that the values of J are never much less than unity and that no term involving either M or N contributes more than 0.007 to the end result. We may thus drop all such terms in both equations (2) and (31) without significant error. The ratio then becomes

$$J = \frac{[1 + u(LZ^2 - K)]^{1/2}}{[1 + u(L - K)]^{3/2}} \left\{ 1 + u(LZ^{-1} - K) - \frac{v}{2qQ}(LZ^{-3} - K) + \frac{3w}{4q^2Q^2}(LZ^{-5} - K) \right.$$

$$\left. + \frac{3U}{4Q^2} + \frac{3Uu}{4Q^2}(LZ^{-5} - K) - \frac{15Uv}{8qQ^3}(LZ^{-7} - K) + \frac{105Uw}{16q^2Q^4}(LZ^{-9} - K) \right\}, \tag{32}$$

where we have set $R \equiv QZ^2$ and where all other quantities have been previously defined.

In table 1 the values of J are computed from equation (32) for several perigee heights, eccentricities, and orientations of the orbit relative to the sunward bulge. The values are given to three decimal places to indicate better the trend of the numbers. Two points should be kept in mind, however, when the entries are examined. First, the final calculations have been much simplified by leaving out all terms in M and N. Second, the convergence of the terms in equation (32) is very rapid in all situations except where the perigee height is large ($z_q \cong 600$ km) and the eccentricity rather small ($e \cong 0.1$). In this extreme situation the atmospheric distortion is a maximum, the scale height is large, and at very low eccentricities the drag may be significant at all points around the orbit. Calculating and integrating the terms in $y^6$ in equation (31), I find that the true value of J in the most extreme case of table 1 is even more exaggerated than the tabulated entry of 1.52; it is greater by about 3 percent. Truncation of equation (29) at order $y^4$ affects other entries by smaller amounts.

## 7. Interpretation of the Results

Generally, the entries of table 1 show that the secular acceleration in a bulging atmosphere of increasing scale height is greater than in the same orbit when the atmosphere is spherically symmetric and of constant scale height. This means that use of the simpler formula tends to produce overestimates of the values of $\rho\sqrt{H}$ from the observed accelerations, although not always.

The dominant reason for $J > 1$ when either the orbit normal or apogee coincides with the bulge is the increase of scale height with height. Reference is made, for these and subsequent remarks, to figure 3 of Jacchia's paper (Jacchia, 1960a), which is helpful in visualizing these effects. When the normal to the orbit plane coincides with the bulge axis the satellite is entirely ignorant of the bulge because it always moves 90° away from it; when apogee coincides with the bulge axis J is not significantly different from unity because of the $\cos^6(\psi'/2)$ dependence. For both orientations the values of J increase with increasing perigee height because of the gradient of the scale height.

When perigee coincides with the bulge, the gradient of the scale height governs the run of J at low values of $z_q$; the bulge is not pronounced at these levels. At 500 and 600 km, however, the interplay of the two effects is stronger. At the higher eccentricities a satellite climbs steeply away from perigee, and in the limiting case of an outbound radial orbit it is oblivious of the bulge and the value of J is governed solely by the effects of scale height. At low eccentricities, however, a satellite climbs uphill from perigee rather slowly and encounters equi-density contours that are trending downhill with increasing distance from the bulge axis. The values of J for these cases are slightly less than unity.

Finally, when the semi-latus rectum of the orbit coincides in direction with the bulge axis, the asymmetry is marked, particularly for large perigee heights and low eccentricities. In this case a satellite encounters maximum density at some point displaced 10° or 15° from the perigee point toward the bulge axis; the secular acceleration is therefore significantly greater, up to 50 or 60 percent, than given by the usual simpler formula.

8. The Equation for a Circular Orbit

A general formula for the drag in a bulging atmosphere can in principle be derived for orbits of small eccentricity, as before, in terms of Bessel functions of imaginary argument. Tentative work indicates, however, that the resulting expression is of formidable complexity and provides little or no insight into the physical effects. Accordingly, this paper will not be concerned with finding an analytic expression for the acceleration in orbits of very small, but non-zero, eccentricity. Qualitatively, however, it is apparent that the more circular an orbit is, the more indifferent the satellite is to the atmospheric density and scale height at perigee. Maximum drag may occur at any point on the orbit.

In the limiting case of a circular orbit, the greatest drag occurs when $\psi'$ is a minimum. The minimum geocentric angle between satellite and bulge axis, $\psi'_{min}$, is given by $\psi'_{min} = \pi/2 - i'$, where $i'$ is the angle between the bulge axis and the orbit normal, with $0 \leqq i' \leqq \pi/2$. The density function to be employed for a circular orbit is given by equation (12) with $s = 0$, and is

$$\rho(0, \psi') = \rho(0, \pi) \{1 + (L - K) \cos^6 (\psi'/2)\}. \tag{33}$$

In this particular case, moreover, $\psi'$ is given by

$$\cos \psi' = \sin i' \cos E , \tag{34}$$

and the secular acceleration turns out to be

$$\frac{\Delta P}{P} = - 3\pi C_D \frac{A}{m} \rho(0, \pi) \left\{1 + \frac{(L - K)}{8} \left[1 + \frac{3}{2} \sin^2 i'\right]\right\}. \tag{35}$$

Since no satellite passes through the point $\psi' = \pi$ unless $i' = \pi/2$, and since all satellites must pass through $\psi' = \pi/2$, we shall find it more useful to express equation (35) in terms of $\rho(0, \pi/2)$, which can be determined from equation (33). The result is then

$$\frac{\Delta P}{P} = - 3\pi C_D \frac{A}{m} \rho(0, \pi/2) \left\{1 + \frac{3(L - K) \sin^2 i'}{16 + 2(L - K)}\right\}. \tag{36}$$

Notice that this result is identical with the simpler equation (7) when $i' = 0$. The reason is, of course, that when $i' = 0$ and $e = 0$ (and under no other conditions) a satellite sees constant density all around the orbit, just as in the spherically symmetric approximation. When $\Delta P/P$ is observed for a satellite moving in a circle with $i' \neq 0$, the atmospheric density deduced from equation (7) will be greater than $\rho(0, \pi/2)$ by a factor equal to the bracket of equation (36). Alternatively interpreted, this factor is the ratio of the secular acceleration at $i'$ to that at $i' = 0$. It is greater than unity because of the sharpness of the bulge: density excesses in the daytime hemisphere more than compensate density deficiencies in the dark hemisphere. When $i' = \pi/2$, the satellite passes squarely through the bulge; in this case the bracket is a maximum, ranging from 1.04 at 200 km to 1.22 at 400 km and up to 1.56 at 600 km.

## 9. Conclusions

It should be emphasized, with Jacchia, that the density function here employed is not a necessary consequence of high-atmosphere physics, but rather a product of numerical analysis of satellite observations. For this reason it does not seem necessary at this stage to work out the exceedingly complex higher-order approximations to equation (32) or the drag equation for orbits of very small but non-zero eccentricity. The results of table 1 for orbits of moderate eccentricity and of section 8 for circular orbits indicate, without more elaborate calculations, that use of the drag formula for a spherically symmetric atmosphere of constant scale height leads to a somewhat erroneous evaluation of the structure of the high atmosphere.

The formulas developed here should, at the expense of some added calculation, make it possible to improve our picture of the Jacchia-type atmosphere by refining the values of its constants. Qualitatively, the results of table 1 indicate that at heights of 500 km or so the atmospheric density on the bulge axis is a bit greater than given by the usual drag formula, while the density 90° around from the bulge axis is substantially less. We may therefore tentatively conclude that the sunward bulge is even sharper than previously thought.

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Table 1

VALUES OF J FOR SEVERAL TYPES OF ORBIT

Perigee coincident with bulge ($\mu = 1$, $\nu = 0$):

| $e$ \ $z_q$(km) | 200 | 300 | 400 | 500 | 600 |
|---|---|---|---|---|---|
| 0.10 | 1.024 | 1.016 | 1.005 | 0.990 | 0.977 |
| 0.15 | 1.030 | 1.028 | 1.026 | 1.022 | 1.018 |
| 0.20 | 1.033 | 1.035 | 1.037 | 1.040 | 1.044 |
| 0.25 | 1.035 | 1.039 | 1.045 | 1.052 | 1.060 |
| 0.30 | 1.035 | 1.041 | 1.050 | 1.060 | 1.072 |

Semi-latus rectum of orbit coincident with bulge ($\mu = 0$, $\nu = \pm 1$):

| $e$ \ $z_q$(km) | 200 | 300 | 400 | 500 | 600 |
|---|---|---|---|---|---|
| 0.10 | 1.044 | 1.076 | 1.145 | 1.280 | 1.520 |
| 0.15 | 1.040 | 1.067 | 1.122 | 1.230 | 1.435 |
| 0.20 | 1.039 | 1.062 | 1.109 | 1.203 | 1.382 |
| 0.25 | 1.038 | 1.059 | 1.101 | 1.185 | 1.346 |
| 0.30 | 1.037 | 1.057 | 1.096 | 1.173 | 1.322 |

Orbit normal coincident with bulge ($\mu = 0$, $\nu = 0$):

| $z_q$(km) | 200 | 300 | 400 | 500 | 600 |
|---|---|---|---|---|---|
| All e | 1.032 | 1.042 | 1.057 | 1.084 | 1.132 |

Apogee coincident with bulge ($\mu = -1$, $\nu = 0$):

| $z_q$(km) | 200 | 300 | 400 | 500 | 600 |
|---|---|---|---|---|---|
| All e | 1.030 | 1.035 | 1.040 | 1.045 | 1.049 |

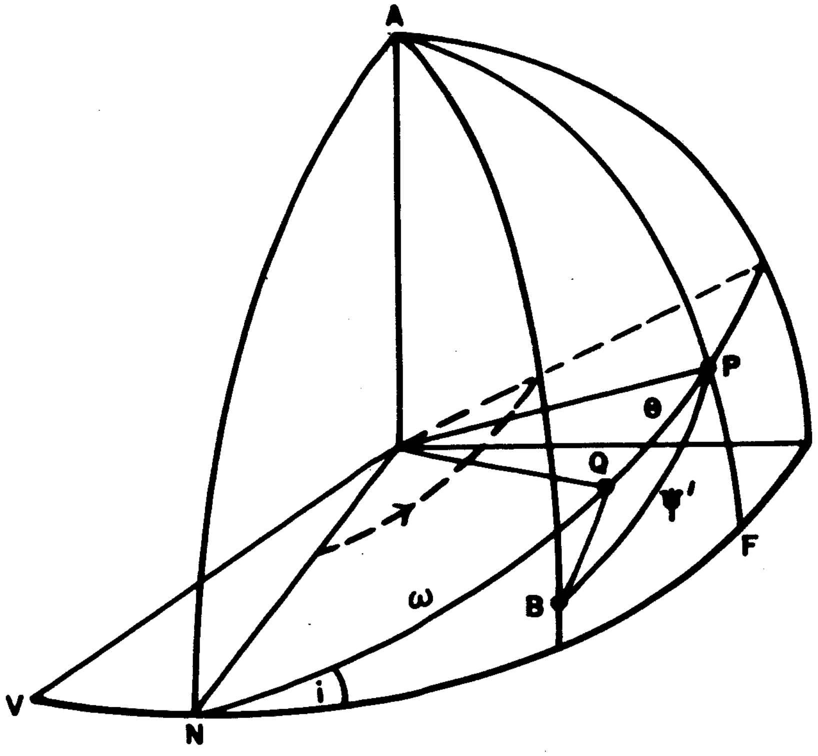


Figure 1. --The celestial sphere, showing the interrelationship of the locations of the atmospheric bulge axis, B, of satellite perigee point, Q, and of the instantaneous satellite position, P.

NOTICE

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